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A Unified Framework for the Kondo Problem and for an Impurity in a Luttinger Liquid

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We develop a unified theoretical framework for the anisotropic Kondo model and the boundary sine-Gordon model. They are both boundary integrable quantum field theories with a quantum-group spin at the boundary which takes values, respectively, in standard or cyclic representations of the quantum group $SU(2)_q$. This unification is powerful, and allows us to find new results for both models. For the anisotropic Kondo problem, we find exact expressions (in the presence of a magnetic field) for all the coefficients in the Anderson–Yuval perturbative expansion. Our expressions hold initially in the very anisotropic regime, but we show how to continue them beyond the Toulouse point all the way to the isotropic point using an analog of dimensional regularization. The analytic structure is transparent, involving only simple poles which we determine exactly, together with their residues. For the boundary sine-Gordon model, which describes an impurity in a Luttinger liquid, we find the non-equilibrium conductance for all values of the Luttinger coupling. This is an intricate computation because the voltage operator and the boundary scattering do not commute with each other.

KEY WORDS: Quantum impurity; Luttinger liquid; Kondo problem; integrable.

1. INTRODUCTION

One-dimensional quantum field theories with gapless bulk excitations and boundary interactions display a wide range of interesting characteristics. They exhibit crossovers between Fermi-liquid and non-Fermi-liquid behavior, they can be successfully treated by a variety of powerful and interesting techniques, and they can be realized experimentally.

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The classic example of such a system consists of electrons interacting with dilute impurities in a metal, which can be described by the Kondo model. This system is actually three-dimensional; it can be described by a one-dimensional model because with dilute enough impurities, the interesting physics occurs in s waves around each impurity, and one can restrict attention to the radial coordinate. This model has a variety of experimental realizations, and has been the focus of much attention in the last 30 years (see refs. 1 and 2 and references therein). Of recent interest is the problem of an impurity in a Luttinger liquid.[3] A Luttinger liquid (interacting electrons in one dimension) may be realized in a one dimensional wire or by the edge of a fractional quantum Hall device.[4] A fractional quantum Hall device is made by putting an electron gas trapped in two dimensions into a strong transverse magnetic field. When the Hall conductivity is locked to its plateau value, the current flows only along the edges of the device, and the system can be described effectively by a one-dimensional theory. Experiments have been done on the conductance through a point contact (which is the impurity in the theory) in one of these devices[5] and they agree well with theory.[6,7]

The objects of our attention in this paper are one-dimensional models with interaction on the boundary only. We concentrate on two such models, the one-channel Kondo model and the massless boundary sine-Gordon model. The problem of an impurity in a Luttinger liquid can be mapped onto the latter. Moreover, when the bulk degrees of freedom are integrated out, both describe problems in dissipative quantum mechanics:[8],[2] a particle moving in a double well for Kondo (an infinite number of wells for boundary sine-Gordon) with a dissipative environment.

In this work, we show that the Kondo model and the boundary sine-Gordon model can be treated in the same theoretical framework. Both can be reformulated as a free boson on the half-line interacting with a spin on the boundary, where the spin is in a representation of the "quantum-group" algebra $SU(2)_q$. This algebra, as we will discuss below, is a one-parameter deformation of the ordinary $SU(2)$ algebra. In the Kondo model the spin is in a standard spin-j representation, while for the boundary sine-Gordon model the spin is in a "cyclic" representation, a quantum-group representation which has no analog in ordinary $SU(2)$. We will find a simple relation between the partition functions of the two models. The relation is established through the use of the trace of the quantum monodromy operator, an object generating the conserved charges of the quantum KdV system.[10] Having this relation, we can relate quantities in one model to quantities in the other model. For example, the perturbative coefficients of

[2] For subsequent developments see ref. 9.

the partition function of the spin-1/2 Kondo model are expressed in terms of ordered integrals which are difficult to evaluate. This relation yields an expression for these coefficients in terms of known coefficients of the boundary sine-Gordon model.[11]

The starting point of the theoretical analysis is a one-dimensional quantum theory at a fixed point of the renormalization group. This means that the system has no mass scale, so there is no gap in the spectrum. Such a model can be described by a $(1+1)$-dimensional massless quantum field theory. The issue of the boundary conditions in these models is not a nuisance, but in fact can be of crucial importance. Basically, most of the physics which can happen in the bulk can also happen on the boundary alone. Studying boundary behavior is not only simpler mathematically, but it can also be easier to observe experimentally. There are very few experimental probes of one-dimensional quantum systems, and the ones mentioned above are both boundary effects.

A boundary fixed point is a point where the boundary condition does not destroy the scale invariance of the bulk; the methods of boundary conformal field theory are applicable here. However, an interacting boundary condition as in both the above systems will introduce a scale to the problem, which we generically call T_B (in the Kondo problem this is often referred to as the Kondo temperature T_K). Although by definition bulk effects in these models do not depend on this scale, boundary effects of a system at non zero temperature can now depend on the dimensionless parameter T/T_B. Varying this parameter allows one to interpolate between different boundary fixed points. For example, in the Kondo problem at $T/T_B \to \infty$, there is a boundary fixed point where the impurity decouples. As $T/T_B \to 0$, one approaches another boundary fixed point where the electrons bind to the impurity. (The properties of the low-temperature fixed point are far from obvious; it took years of effort to establish them.) For the boundary-sine Gordon model, the fixed points correspond to Dirichlet and Neumann boundary conditions on a boson; which is the high-temperature one and which is the low-temperature one depends on the boundary coupling.

The field theories we discuss have the special property that they are integrable, as are many one-dimensional theories. As a result, we can do many calculations exactly. In this paper, we mainly discuss the partition function and free energy. However, transport properties (which are experimentally measurable) can also be computed exactly.[7, 12] The methods we will describe enable one to study these systems for all values of the coupling—near and far from the fixed points. Other methods generally rely on perturbation theory around these fixed points. Another advantage, for example, is that in the Luttinger problem one can compute transport properties such as the conductance even out of equilibrium. Standard field theory techniques are not applicable; the best one can do is use the Kubo formula to calculate the linear response near $V=0$.

The models discussed here can be treated as a boson where the one-dimensional space is a half-line. The only interactions take place on the boundary. The bulk Hamiltonian takes the form

$$H_0=\frac{1}{4\pi g}\int_0^{\infty} d\sigma\,[\Pi^2+(\partial_\sigma\phi)^2]+\frac{V}{2\pi}\int_0^{\infty} d\sigma\,\partial_t\phi \tag{1.1}$$

The first model we discuss is the boundary sine-Gordon model. The boundary Hamiltonian is

$$H_{\mathrm{BSG}}=2v\cos\phi(0) \tag{1.2}$$

In this case, the parameter V of (1.1) plays the role of a physical voltage, while v is related to the boundary scale T_B in a manner to be discussed below.

The second model, the one-channel anisotropic Kondo problem of spin $j/2$, can also be expressed in this form using the well-known technique of bosonization.[13] We ignore the charge sector of the Hamiltonian, which does not interact with the spin and decouples from the problem. The total Hamiltonian is then $H=H_0+H_j$, where the boundary interaction is $H_j=\sum_{i=x,y,z} I_i J_i S_i$. Here, S_i is the impurity spin on the boundary, J_i are the fermion currents, and I_i are the coupling constants. The problem is anisotropic when $I_z\neq I_x=I_y$. In the bosonized language, the boundary Hamiltonian becomes

$$H_j=\lambda(S_+e^{i\phi(0)}+S_-e^{-i\phi(0)}) \tag{1.3}$$

We have replaced the original parameters $I_x=I_y$ and I_z with g and λ; g parametrizes the anisotropy ($g=1$ is the isotropic case and $g=1/2$ is called the Toulouse limit), while $\lambda\propto|I_x|$. The precise relation is not universal, so we will not need it here. The J_zS_z term has been absorbed in a redefinition of g. Traditionally in the Kondo problem, one takes the matrices S_i to act in the spin-$j/2$ representation of $SU(2)$ [in (1.3) and in all that follows we use the convention that eigenvalues of S_z are integer, so they, e.g., take values $S_z=\pm 1$ in the spin-1/2 representation]. However, for the problem to be integrable, one must instead take them to act in the spin-$j/2$ representation of the quantum group $SU(2)_q$,

$$[S_z,S_\pm]=\pm 2S_\pm,\qquad [S_+,S_-]=\frac{q^{S_z}-q^{-S_z}}{q-q^{-1}} \tag{1.4}$$

where $q = e^{i\pi g}$. In the isotropic case $q = -1$, this reduces to the usual $SU(2)$ algebra. The distinction between $SU(2)$ and $SU(2)_q$ is not important for $j = 1$ or $j = 2$ at arbitrary q, because the spin-1/2 representation remains the Pauli matrices and the spin-1 representation is also the same up to a rescaling of S_+ and S_-. In the following, the Kondo model of spin $j/2$ is defined to be the model with the q-deformed algebra, so it can be identified with the "physical" Kondo model only for $j = 1, 2$. In the Kondo models V corresponds to an external magnetic field.

We consider the system in imaginary time compactified on a circle of length $1/T$, with T the temperature. Defining the partition function via the trace $\mathscr{Z}_j = \mathrm{Tr}\exp[e^{i\pi p S_z}(H_0 + H_j)/T]$, we introduce $Z_j = \mathscr{Z}_j(\lambda)/\mathscr{Z}_j(0)$ and $Z_{\mathrm{BSG}} = \mathscr{Z}_{\mathrm{BSG}}(\lambda)/\mathscr{Z}_{\mathrm{BSG}}(0)$. In the following, we often use the variable p defined as

$$i\frac{Vg}{T} = 2\pi p \tag{1.5}$$

Most of the following computations are well defined only when p is an integer. The consideration of real (physical) voltage or magnetic field requires analytic continuation, which we discuss.

The dimension of the vertex operators $e^{\pm i\phi}$ is g. For example the two-point function on the boundary is (calling τ the imaginary time)

$$\langle e^{i\phi(0,\tau)} e^{-i\phi(0,\tau')} \rangle = \left| \frac{\kappa}{\pi T} \sin \pi T(\tau - \tau') \right|^{-2g} \tag{1.6}$$

with κ the frequency cutoff arising from the normal ordering of the operators $e^{\pm i\phi}$. We will denote the case $1/2 < g < 1$ as the repulsive regime and $0 < g < 1/2$ the attractive regime; "attractive" and "repulsive" are the corresponding types of fermion interactions when one fermionizes this model into the Luttinger model.[3] The Toulouse limit $g = 1/2$ corresponds, of course, to free fermions. For $g > 1$ the vertex operators are irrelevant, and the model is best approached by using a "dual" picture.[(3)]

The paper is organized as follow. In Section 2, the attractive regime is described. Results are obtained to all orders in perturbation theory using Jack symmetric functions for the boundary sine-Gordon model. Then, making use of the monodromy matrix, we give a relation between this

[3] In the quantum wire problem (an impurity in a Luttinger liquid) where one starts with electrons on a full line, the entire domain $0 < g < 1$ corresponds to repulsive interactions between the physical electrons. There is a rescaling of the coupling when one maps the model onto the half-line.[(7)]

latter model and the Kondo models. In the last part of this section, the thermodynamic Bethe ansatz is used to provide nonperturbative results in both cases. In Section 3, the repulsive regime is explored. There the perturbative coefficients of the partition function diverge and a regularization is needed, which usually is provided by a high-frequency or short-distance cutoff. We show, using the explicit expressions discussed in Section 2, that these divergences can also be controlled by analytic continuation from the repulsive regime, an analog of dimensional regularization. Coefficients for the free energy can be obtained in this fashion all the way to $g = 1$; at particular values of g there are poles, and we compute the residues exactly. These results are in agreement with computations using the Bethe ansatz in the repulsive regime, where the poles result in logarithmic terms in the free energy. In Section 4, the relation between models is extended to nonzero V. This gives the perturbative coefficients in Kondo model as a function of magnetic field. Moreover, it yields the previously unknown conductance for the boundary sine-Gordon model at all values of g. Some final remarks are collected in the conclusion.

2. THE ATTRACTIVE REGIME AT ZERO VOLTAGE

In this section we review earlier results for the anisotropic Kondo problem at zero magnetic field and the boundary sine-Gordon (BSG) model at zero voltage and with $g < 1/2$. There are three useful and complementary approaches, all of which we will later extend to finite magnetic field (resp. finite voltage) and to $g > 1/2$.

We first discuss how to expand the partition function in powers of the interaction strength. For the Kondo problem, this was first considered long ago in refs. 14 and 13, where the coefficients of this expansion were expressed as multiple integrals. These integrals are rather complicated, and until now had not been evaluated explicitly except in very special limits. The partition function in this form is equivalent to that of a one-dimensional gas of positive and negative charges with logarithmic interactions (equivalently of a two-dimensional Coulomb gas on a circle). For the boundary sine-Gordon model, the perturbative expansion is formally very similar, but not identical. In that case, the multidimensional integrals can be explicitly evaluated, using recent results for symmetric polynomials.[11] We note that this is the only known perturbative expansion for the Kondo problem. Similar-looking expansions have been employed in the Anderson model, but even though the Anderson model reproduces the Kondo model in a particular limit, these expansions are applicable only to the Anderson model very far from the Kondo limit.

The second approach uses integrability, albeit in a rather abstract way. We define the trace of the "quantum monodromy operator,"[10] whose expectation value gives the Kondo or BSG partition function, depending on which representation is chosen.[15, 16] Using some properties of this operator, we are able to relate the Kondo partition function to the BSG one. As a result we are able to evaluate the integrals in the Kondo expansion explicitly, using the already evaluated BSG ones.[15, 16]

The third approach uses integrability in a more standard way. We describe the model in terms of interacting quasiparticles and their scattering matrices. The thermodynamic Bethe ansatz (TBA) can then be used to derive the free energy and related quantities for the Kondo model[1, 2] and for the BSG model.[17] The direct relation between the partition functions can be rederived, at least for values of the coupling $g = 1/t$, t integer. The TBA approach has the disadvantage that at nonzero temperature the integral equations derived are not continuous in g (although the final results of course are). However, it has the advantage that it allows transport properties such as the current and conductance[7, 12] and the zero-temperature noise[18] to be computed for the BSG model. (For Kondo, only the zero-temperature magnetoresistance has been computed.[1]) Some simple relations have also been derived relating transport properties to equilibrium properties;[11] we generalize these in Section 4.

2.1. Perturbative Approach

The partition functions Z_j and Z_{BSG} can be expanded in powers of λ and v, respectively. The term of order λ^{2n} or v^{2n} involves a correlation function of n vertex operators $e^{i\phi}$ and n vertex operators $e^{-i\phi}$, all living on the boundary. These multipoint functions, evaluated in the free-boson theory and by Wick's theorem, are reduced to a product of two-point functions like (1.6) (see, e.g., ref. 19 for a review). The problem then becomes formally equivalent to a two-dimensional Coulomb gas with positive and negative charges restricted to live on a one-dimensional circle. To calculate the partition function, we must integrate over the locations of the charges. The integrand is then the scaled correlator

$$
\begin{aligned}
&\mathscr{I}_{2n}(\{u_i\}, \{u'_i\}) \\
&\equiv \left(\frac{\kappa}{2\pi T}\right)^{2gn} \left\langle e^{i\phi(u_1/2\pi T)} \cdots e^{i\phi(u_n/2\pi T)} e^{-i\phi(u'_1/2\pi T)} \cdots e^{-i\phi(u'_n/2\pi T)} \right\rangle \\
&= \left| \frac{\prod_{i<j} 4\sin((u_i - u_j)/2)\sin((u'_i - u'_j)/2)}{\prod_{i,j} 2\sin((u_i - u'_j)/2)} \right|^{2g} \qquad (2.1)
\end{aligned}
$$

The difference between the Kondo model and the boundary sine-Gordon (BSG) model lies in the limits of integration. In the Kondo model, each of the vertex operators comes with a spin operator; the thermal average of monomials of vertex operators are computed as in (2.1), while one has to take the trace of the corresponding monomial of spin operators in the representation of interest. This puts various contraints on the order of the charges. For example, $S_+^2 = S_-^2 = 0$ when the spin is 1/2, so only terms of the form $S_+ S_- S_+ S_- \cdots$ survive in the perturbative expansion, and consequently charges alternate in sign on the circle. Thus in terms of the renormalized parameter x, defined as

$$x \equiv \frac{\lambda}{T}\left(\frac{2\pi T}{\kappa}\right)^g$$

the spin-1/2 Kondo partition function is[13,14]

$$Z_1(x) = 2 + \sum_{n=1}^{\infty} x^{2n} Q_{2n} \tag{2.2}$$

where

$$Q_{2n}(p) = 2\int_0^{2\pi} du_1 \int_0^{u_1} du'_1 \int_0^{u'_1} du_2 \cdots \int_0^{u_n} du'_n \, \mathscr{I}_{2n}(\{u_i\}, \{u'_i\}) \tag{2.3}$$

The effect of the charge ordering is seen in the limits of integration. Higher-spin partition functions have the same integrand, but with the appropriate restrictions on charge ordering. For the boundary sine-Gordon model there is no boundary spin, so one has unordered integrals:

$$Z_{\mathrm{BSG}}(x) = 1 + \sum_{n=1}^{\infty} x_{\mathrm{BSG}}^{2n} I_{2n} \tag{2.4}$$

where

$$x_{\mathrm{BSG}} \equiv \frac{v}{T}\left(\frac{2\pi T}{\kappa}\right)^g$$

and

$$I_{2n} = \frac{1}{(n!)^2}\int_0^{2\pi}\int_0^{2\pi} du_1 \cdots \int_0^{2\pi} du'_n \, \mathscr{I}_{2n}(\{u_i\}, \{u'_i\}) \tag{2.5}$$

The lack of ordering makes no difference for $n=1$, so $I_2 = Q_2$, but the others are different.[4]

[4] The I_{2n} were denoted Z_{2n} in refs. 11 and 15; we change notation here to avoid confusion with the higher-spin partition functions.

The unordered integrals I_n can be computed exactly in terms of an n-dimensional series[11]:

$$I_{2n} = \frac{1}{[\Gamma(g)]^{2n}} \sum_{\mathbf{m}} \prod_{i=1}^{n} \left(\frac{\Gamma[m_i + g(n-i+1)]}{\Gamma[m_i + g(n-i)+1]} \right)^2 \tag{2.6}$$

where the sum is over all sets (Young tableaux) $\mathbf{m} = (m_1, m_2, ..., m_n)$, with integers m_i obeying $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$. For $n=1$ the series can be summed, giving $I_2 = \Gamma(1-2g)/[\Gamma(1-g)]^2$. Although this series looks quite imposing, it can be generated by a simple recursion relation. We introduce the truncated sum $I_{2n}(\Lambda)$, which is defined as the sum over $\mathbf{m}$ with the condition that all $m_i \leqslant \Lambda$. Then it is not difficult to show that

$$I_{2n}(\Lambda) = I_{2n}(\Lambda - 1) + \left(\frac{\Gamma(\Lambda + gn)}{\Gamma(g)\, \Gamma(\Lambda + 1 + g(n-1))} \right)^2 I_{2(n-1)}(\Lambda) \tag{2.7}$$

These relations allow a precise determination of the partition functions up to large orders in the perturbation expansion.

The boundary sine-Gordon model can be placed in the same framework as the anisotropic Kondo models. It is enough to discuss the simplest case when q is a root of unity (g rational), since, by continuity, the results we derive will hold for any q of unit modulus. Suppose therefore $q^k = \pm 1$ for some integer k, and consider a cyclic representation of the quantum group $SU(2)_q$.[20] These representations, which are labeled by an arbitrary complex parameter δ, have no highest- or lowest-weight state; the states are eigenstates of S_+ or S_- to the tth power. They have dimension k, with a basis of states $|m\rangle$ such that

$$\begin{aligned} S_+ \, |m\rangle &= \frac{q^{\delta - m/2} - q^{-(\delta - m)/2}}{q - q^{-1}} \, |m+1\rangle \\ S_- \, |m\rangle &= \frac{q^{\delta + m/2} - q^{-(\delta + m)/2}}{q - q^{-1}} \, |m-1\rangle \\ S_z \, |m\rangle &= m \, |m\rangle \end{aligned} \tag{2.8}$$

where states $|m\rangle$ and $|m \bmod k\rangle$ are identified, and the fundamental set is chosen to be 0, 1,..., $(k-1)$. To obtain the BSG model, we set $q^\delta = C$ and let $C \gg 1$ and real (so δ is imaginary). Thus

$$S_\pm \, |m\rangle \approx C \frac{q^{\mp m/2}}{q - q^{-1}} \, |m \pm 1\rangle \tag{2.9}$$

so in this limit the commutator of $S_\pm$ can be neglected and the traces of all monomials become identical:

$$\mathrm{Tr}\,\mathcal{M} = k\left(\frac{Cq^{1/2}}{q-q^{-1}}\right)^{2n} \tag{2.10}$$

where $\mathcal{M}$ is the product of n operators S_+ and n operators S_- in any order. Thus when evaluating Z_δ for Clarge, all the possible orderings of S_+ and S_- within the trace have the same weight, so

$$Z_\delta(x) \approx kZ_{\mathrm{BSG}}\left(C\frac{q^{1/2}}{q-q^{-1}}x\right), \qquad q^\delta = C \gg 1 \tag{2.11}$$

This observation allows us, for example, to find the boundary S matrix of the BSG model,[21] as we detail in the Appendix. It will also enable us to derive many properties of the partition function Z_{BSG} in the subsequent sections.

2.2. Quantum Monodromy and Fusion

We introduce the quantum monodromy operators associated with these models[10]

$$\begin{aligned} L_j(x) = \Pi_j\Bigg\{ e^{i\pi P_L S_z}\mathcal{P}\exp\Bigg[q^{-1/2}x\int_0^{1/T} d\tau \\ \times\left(e^{-2i\phi_L(\tau)}q^{S_z/2}S_+ + e^{2i\phi_L(\tau)}q^{-S_z/2}S_-\right)\Bigg]\Bigg\} \end{aligned} \tag{2.12}$$

where Π_j indicates that the matrices S_j are in the spin-$j/2$ representation, $\mathcal{P}$ indicates path ordering, and the exponentials are normal-ordered. In this formula, P_L is the momentum operator appearing in the mode expansion of the left-moving field ϕ_L,

$$\frac{1}{2\pi g}\phi_L(\tau) = Q_L + P_L\tau + i\sum_{n\neq 0}\frac{a_n}{n}e^{-2i\pi nT\tau}$$

Observe that L_j is an operator acting both on the spin degrees of freedom and on the "free-boson" degrees of freedom. By expanding L_j in powers of x and noting that with Neumann boundary conditions $2\phi_L(0,\tau) = \phi(0,\tau)$, one finds that the partition functions are equal to the eigenvalues of the quantum transfer matrices acting on momentum eigenstates $P\,|p\rangle = p\,|p\rangle$,

$$Z_j(x,p) = \langle p|\,\mathrm{tr}\, e^{i\pi P_L S_z}L_j(x)\,|p\rangle \tag{2.13}$$

where the trace is computed over the spin degrees of freedom. At zero voltage, $p=0$; we define $Z_j(x) \equiv Z_j(x, p=0)$.

As observed in ref. 10, the L_j satisfy the Yang–Baxter equation. Using the fusion of quantum transfer matrices, one can prove the identities

$$\begin{aligned} Z_j(q^{1/2}x)\,Z_j(q^{-1/2}x) &= 1 + Z_{j-1}(x)\,Z_{j+1}(x) \\ Z_1(q^{(j+1)/2}x)\,Z_j(x) &= Z_{j+1}(q^{1/2}x) + Z_{j-1}(q^{-1/2}x) \\ Z_1(q^{(\delta+1)/2}x)\,Z_\delta(x) &= Z_{\delta+1}(q^{1/2}x) + Z_{\delta-1}(q^{-1/2}x) \end{aligned} \tag{2.14}$$

The first was discussed in ref. 10; the second can by proven by using the first and by induction. The last follows using the same technique as in ref. 22 together with the fusion rules for cyclic and standard representations.[23]

Using these relations together with (2.11), we can express the boundary sine-Gordon model partition function in terms of the Kondo partition function. We have, from (2.14),

$$Z_1(Cq^{1/2}x)\,Z_{\text{BSG}}\left(\frac{Cq^{1/2}}{q-q^{-1}}\,x\right) = Z_{\text{BSG}}\left(\frac{Cq^{3/2}}{q-q^{-1}}\,x\right) + Z_{\text{BSG}}\left(\frac{Cq^{-1/2}}{q-q^{-1}}\,x\right)$$

from which it follows that[15,16]

$$Z_1[(q-q^{-1})\,x] = \frac{Z_{\text{BSG}}(qx) + Z_{\text{BSG}}(q^{-1}x)}{Z_{\text{BSG}}(x)} \tag{2.15}$$

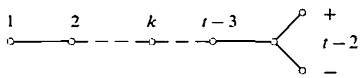
Inserting the perturbative expansions into (2.15) gives the Q_{2n} in terms of the already known I_{2n}, thus completing the derivation of the perturbative partition function for $g<1/2$.

2.3. The Thermodynamic Bethe Ansatz

The fusion relations discussed in the previous subsection are one of the many consequences of integrability.[24] The standard way of approaching the problem is to use the Bethe ansatz. Here, one derives integral equations which determine a set of functions $\varepsilon_j(\theta)$, where θ is a rapidity (the logarithm of the energy of an individual particle). The $\varepsilon_j(\theta)$ can be thought of as the energy of an interacting quasiparticle, in the sense that the energy of the entire system shifts by $T\varepsilon_j(\theta)$ when a particle of rapidity θ is added to the system. Moreover, the distribution function is given by $1/[1+\exp(\varepsilon_j)]$. Many physical quantities can be expressed in terms of these functions. Since this approach has been discussed in detail in many places, we start with the integral equations and discuss their consequences. For technical reasons, we consider the case $g = 1/t$, where t is an integer. The integral equations for both Kondo and BSG are given by[1,2,17]

$$\varepsilon_j = \sum_k N_{jk} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{t-1}{\cosh[(t-1)(\theta-\theta')]} \ln(1+e^{\varepsilon_k(\theta')}) \tag{2.16}$$

where the "incidence matrix" N_{jk} is defined by the diagram



where $N_{jk} = 1$ if the nodes j and k are connected, and $N_{jk} = 0$ if not. The solution of these integral equations is fixed uniquely by demanding the asymptotic form

$$\varepsilon_j \approx 2 \sin \frac{j\pi}{2(t-1)} e^{\theta}, \qquad \varepsilon_\pm \approx e^{\theta} \qquad \text{as} \quad \theta \to \infty \tag{2.17}$$

This asymptotic form is just the energy of the individual particle over T; the interactions become negligible in the large-energy limit (equivalent to sending λ and v to zero in the Hamiltonian).

The free energies in this regime for Kondo[2] (with spin less than $t/2$) and BSG[17] can be written in the form

$$F_j = T_B \frac{\sin[j\pi/2(t-1)]}{\cos[\pi/2(t-1)]} - T \int \frac{d\theta}{2\pi} \frac{t-1}{\cosh[(t-1)(\theta - \ln T_B/T)]} \ln(1+e^{\varepsilon_j}) \tag{2.18}$$

for $j = 1,..., t-2$ together with $F_{t-1} = 2F_{\mathrm{BSG}}$ and

$$F_{\mathrm{BSG}} = \frac{T_B}{2\cos[\pi/2(t-1)]} - T \int \frac{d\theta}{2\pi} \frac{t-1}{\cosh[(t-1)(\theta - \ln T_B/T)]} \ln(1+e^{\varepsilon_{t-1}}) \tag{2.19}$$

where $\varepsilon_{t-1} \equiv \varepsilon_+ = \varepsilon_-$.

It has been shown that Eq. (2.16) requires that $\varepsilon_j(\theta) = \varepsilon_j(\theta + i2\pi/(t-1))$.[25] This means that the integrals in (2.18) and (2.19) can be expanded as a power series in $(T_B/T)^{2(t-1)/t}$, so we see that the bare couplings λ and

v and the renormalized coupling x are proportional to $T_B^{(t-1)/t}$. In fact, for BSG, the exact constant was determined in ref. 12, and is,

$$x_{\mathrm{BSG}} \equiv \frac{v}{T}\left(\frac{2\pi T}{\kappa}\right)^g = \Gamma(g)\left(\frac{T_B}{T}\frac{\Gamma(1/[2(1-g)])}{2\sqrt{\pi}\,\Gamma(g/[2(1-g)])}\right)^{1-g} \tag{2.20}$$

for any value of g, not just $g=1/t$. At fixed T_B, the Kondo bare coupling λ is related to the bare BSG coupling via a constant to be determined at the end of this section: $\lambda \equiv \xi v$. This constant ξ is independent of the impurity spin considered, as observed in ref. 2. With this relation of x and T_B, we see that the second term in (2.18) or (2.19) is an analytic power series in x, like the perturbative partition functions in Section 2.1.

We must take care in relating these nonperturbative free energies to the perturbative partition functions discussed before. The TBA deals with excitations over the vacuum; by convention, the ground state (no particles) is assigned a vanishing energy and entropy. Therefore, one expects F_j and F_{BSG} to be equal to the perturbative partition functions defined previously up to a constant shift (which on dimensional grounds must be proportional to T_B) and a term proportional to T. Neither of these changes, e.g., the specific heat. This ambiguity is fixed by studying the behavior at $T_B=0$ and by studying the analyticity properties. From the TBA equations, it is simple to derive from (2.16) that as $\theta \to -\infty$, the functions $\varepsilon_j(\theta)$ go to a constant, which is

$$e^{\varepsilon_j(-\infty)} = (j+1)^2 - 1, \qquad e^{\varepsilon_\pm(-\infty)} = t-1 \tag{2.21}$$

Plugging this into (2.18) gives

$$F_j(T_B=0) = -T\ln(1+j), \qquad F_{\mathrm{BSG}}(T_B=0) = -\frac{T}{2}\ln t$$

This fixes the piece proportional to T. The piece proportional to T_B is fixed by noticing that because the perturbative partition function is analytic in $x \propto T_B^{1-g}$, the term proportional to T_B cannot appear here. Thus the relation between the perturbative partition functions and the TBA free energies is

$$\begin{aligned} F_j &= -T\ln Z_j + T_B\frac{\sin[j\pi/2(t-1)]}{\cos[\pi/2(t-1)]} \\ F_{\mathrm{BSG}} &= -T\ln Z_{\mathrm{BSG}} + \frac{T_B}{2\cos[\pi/2(t-1)]} - \frac{T}{2}\ln t \end{aligned} \tag{2.22}$$

The role of the shift is simple: it precisely cancels the large T_B/T behavior of the partition functions. As is easily seen by substituting the asymptotic form (2.17) into (2.18) and (2.19), the free energies F_j/T and F_{BSG}/T determined by the TBA go to zero as $T/T_B \to 0$. Meanwhile it was shown in ref. 11 that $-\ln Z_{\mathrm{BSG}} \propto T_B/T$ in this limit, and the relations (2.15) and (2.14) indicate that $-\ln Z_j$ grows as well. In fact from (2.22) it follows that

$$Z_j \approx \exp\left(\frac{T_B \sin[j\pi/2(t-1)]}{T\cos[\pi/2(t-1)]}\right)$$

$$Z_{\mathrm{BSG}} \approx \frac{1}{\sqrt{t}} \exp\left(\frac{T_B}{2T\cos[\pi/2(t-1)]}\right)$$

Thus we see that although the perturbative partition functions grow exponentially for large x, the series expressions are still convergent (they actually have an infinite radius of convergence).

Since we have related the TBA results to the perturbative ones, we can combine the the fusion relations (2.14) with the TBA equations (2.16) to give much more information. For example, the relation (2.18) gives the Kondo partition functions Z_j only for $j = 1,\ldots, t-2$, but the remainder can be generated from (2.14). The relation (2.22) allows us to write the perturbative partition functions in terms of the ε_j very simply. Denoting convolution by

$$A * B(\alpha) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} A(\alpha - \theta)\, B(\theta)$$

one has

$$\begin{aligned} \ln Z_j(x) &= s_{t-1} * \ln(1 + e^{\varepsilon_j})(\alpha), \qquad j = 1,\ldots, t-2 \\ \ln Z_{\mathrm{BSG}}(\xi x) &= -\tfrac{1}{2}\ln t + s_{t-1} * \ln(1 + e^{\varepsilon_{t-1}})(\alpha) \qquad (2.23) \\ \ln Z_{t-1}(x) &= 2s_{t-1} * \ln(1 + e^{\varepsilon_{t-1}})(\alpha) \end{aligned}$$

where $s_a = a/\cosh(a\theta)$ and $\alpha = \ln T_B/T$. These relations also give immediately

$$Z_{t-1}(x) = tZ^2_{\mathrm{BSG}}(\xi x) \qquad (2.24)$$

In the following, we often switch from the x variable to α, keeping these relations in mind. Introduce as usual

$$Y_j(\alpha) = e^{\varepsilon_j(\alpha)} \qquad (2.25)$$

The Y_j are analytic functions of $x^2 \propto \exp[2t\alpha/(t-1)]$.[25] From the TBA equation for ε_1 one finds then $Y_1(\alpha)=Z_2(x)$ and from the TBA relation for ε_{t-1}, $Y_{t-1}(\alpha)=Z_{t-2}(x)$. The TBA relations for the other nodes then imply that

$$Y_j(\alpha)=Z_{j+1}(x)\,Z_{j-1}(x), \qquad j=1,\dots,t-2 \tag{2.26}$$

This simple relation between the TBA and the perturbative partition function gives exact perturbative expressions for all the ε_j. This is consistent with the relation (2.24), since the original integral equations (2.16) along with (2.23) give $Y_{t-2}=tZ_{t-3}Z^2_{\mathrm{BSG}}$.

We can in fact rederive the results of Section 2.2 at $q=e^{i\pi/t}$ by converting the TBA equations (2.16) into functional equations in the complex a plane. Using the identity

$$s_a\left(\theta+i\frac{\pi}{2a}\right)+s_a\left(\theta-i\frac{\pi}{2a}\right)=2\pi\,\delta(\theta) \tag{2.27}$$

with (2.23) gives

$$Z_j(q^{1/2}x)\,Z_j(q^{-1/2}x)=1+Y_j(\alpha) \tag{2.28}$$

where $j=1,\dots,t-2$. Plugging the relation (2.26) into (2.28) recovers the fusion relation (2.14)

$$Z_j(q^{1/2}x)\,Z_j(q^{-1/2}x)=1+Z_{j-1}(x)\,Z_{j+1}(x) \tag{2.29}$$

from ref. 9. Once written in terms of the variable q, (2.29) holds for q generic, as discussed in Section 2.2. Observe that such a direct proof of (2.29) establishes conversely the fact that the relation between T_B/T and x is independent of spin.

When $q=e^{i\pi/t}$, t an integer, these fusion relations close.[5] We have from (2.29)

$$Z_{t-1}(q^{1/2}x)\,Z_{t-1}(q^{-1/2}x)=1+Z_{t-2}(x)\,Z_t(x)$$

while using (2.27) in (2.23) gives

$$Z_{\mathrm{BSG}}(\xi q^{1/2}x)\,Z_{\mathrm{BSG}}(\xi q^{-1/2}x)=\frac{1}{t}\left[1+Y_{t-1}(\alpha)\right] \tag{2.30}$$

[5] A different closure happens in the minimal models of conformal field theory,[10] where the end nodes $t-2$, $+$ and $-$ are removed from the incidence diagram.

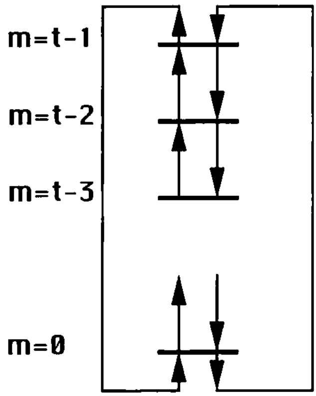


Fig. 1. A schematic representation of a cyclic representation of $SU(2)_q$ when q is a tth root of unity. Up and down arrows represent the action of raising and lowering generators, respectively.

Thanks to the identification $Y_{t-1} = Z_{t-2}$ these two relations are compatible with (2.24) if and only if

$$Z_t(x) = Z_{t-2}(x) + 2 \tag{2.31}$$

The latter relation follows from the quantum group representation. Indeed, when q is a tth root of unity, the representation of spin t is reducible because $S'_{\pm} = 0$, and looks schematically as in Fig. 1. We see that the states with values $S_z = \pm t$ do not contribute to the trace of any monomial in $S_+ S_-$ of non vanishing order. Hence in the perturbative expansion, all terms for spin $t-2$ and spin t are equal, except the term of order zero, which simply counts the number of states. This term differs by two in the two representations, and (2.31) follows.

We can also rederive the relation (2.15) between the Kondo and BSG partition functions without using cyclic representations, by using the fusion relation from (2.14):

$$Z_1(ix)\, Z_{t-1}(x) = Z_{t-2}(q^{-1/2}x) + Z_t(q^{1/2}x)$$

Using (2.31), we rewrite the right-hand side as

$$1 + Z_{t-2}(q^{1/2}x) + 1 + Z_{t-2}(q^{-1/2}x)$$

which, using $Y_{t-1} = Z_{t-2}$ and (2.30), is in turn

$$tZ_{\mathrm{BSG}}(\xi x)[Z_{\mathrm{BSG}}(\xi q x) + Z_{\mathrm{BSG}}(\xi q^{-1} x)]$$

from which, using (2.24), it follows that

$$Z_1[(q-q^{-1})x] = \frac{Z_{\mathrm{BSG}}(qx) + Z_{\mathrm{BSG}}(q^{-1}x)}{Z_{\mathrm{BSG}}(x)} \tag{2.32}$$

together with the fact that $\xi = i/(q-q^{-1})$. This value for ξ is obtained simply by matching the first order in the perturbation theory, since we know that $I_2 = Q_2$.

3. THE REPULSIVE REGIME AT ZERO VOLTAGE

3.1. Poles and Log Terms in the Perturbative Expansion

The previous section concerned the attractive regime $g < 1/2$. This, for example, is the regime of greatest interest in dissipative quantum mechanics, where the particle exhibits oscillatory behavior. The filling fractions $\nu = 1/(2n+1)$ where the edge modes in the fractional quantum Hall effect are described by the BSG model also lie in this regime. However, the original isotropic Kondo model is at $g = 1$, and we will see that there is a great deal of interesting behavior in the repulsive regime $1/2 < g < 1$. We will address this regime largely by exploiting some simple analyticity properties. In particular, we will show how to obtain an analytic expression for the coefficients all the way to $g = 1$.

We will show in this section that if we define the expansion of the free energy of the spin-1/2 Kondo model as

$$-T \ln Z_1 = T \sum_{n=0}^{\infty} f_{2n} x^{2n} = -T \ln \left(2 + \sum_{n=1}^{\infty} Q_{2n} x^{2n} \right)$$

the term $f_{2n} x^{2n}$ has a simple pole at $g = 1 - 1/(2n)$, with residue $r_{2n} = T_B/(2\pi T n^2)$. The half-integer-spin Kondo and BSG free-energy expansions have poles in the same places. Moreover, when this divergence is regulated properly, we find a term $-2nr_{2n} T \log(T_B/T)$ in the free energy for $g = 1 - 1/(2n)$. This term yields, for example, a term linear in T_B/T in the specific heat, indicating that these values of g (which include the Toulouse limit $g = 1/2$) are pathological in some respects.[6]

[6] However, notice that now a term $T^{2n(1-g)}$ appears in the specific heat expansion at all g; the log term is required to make this true at $g = 1 - 1/(2n)$. Thus in this sense the log terms are not pathological, but instead make the values $g = 1 - 1/(2n)$ *more* like other values of g.

The first thing to notice is that the integrals (2.3) and (2.5), which define the perturbative coefficients I_{2n} and Q_{2n} diverge at short distances when $g \geqslant 1/2$; correspondingly, the series expansion (2.6) diverges for $g \geqslant 1/2$. There are a variety of ways to regulate the integrals. In a numerical approach, this would be done using a cutoff. However, the most natural approach here is analytic continuation. This approach, which is very analogous to dimensional regularization,[26] means we *define* the regularized integrals as the analytic continuation of their values for $g < 1/2$. This continuation is illustrated by examing the first coefficient, $f_2 = -I_2/2$, which we saw in Section 2.1 is given by $I_2 = \Gamma(1-2g)/[\Gamma(1-g)]^2$. At $g = 1/2$, f_2 has a simple pole. There are no branch points anywhere, and since it is finite for all other $g \leqslant 1$, the analytic continuation is perfectly well defined. Implicit in the following is the assumption that the regularization done for the Bethe ansatz (the cutoff of the Fermi sea for Kondo[2]) gives the same results as this analytic continuation from $g < 1/2$. This assumption is certainly physically obvious, since by defining renormalized parameters one removes all cutoff dependence from the Bethe ansatz. Moreover, in the BSG model one starts directly from the regulated theory with no cutoff dependence.[21]

We now show that the large-T_B/T behavior of the partition function requires that $f_{2n}^{(\mathrm{BSG})}$ and f_{2n} have simple poles at $g = 1 - 1/(2n)$ for all n. As discussed in Section 2.3, in this limit $-T \ln Z_{\mathrm{BSG}}$ behaves like

$$-T_B \frac{1}{2\cos(\pi g/[2(1-g)])}$$

while $-T \ln Z_1$ goes as

$$-T_B \tan\left(\frac{\pi g}{2(1-g)}\right)$$

Our analyticity assumption implies that these hold for $g > 1/2$ as well. Notice that these expressions have a simple pole as $g \to 1 - 1/(2n)$. In the TBA free energy this term is subtracted off, as seen it (2.22). Because the TBA is finite, this divergence therefore is matched by one in the perturbative expansion. Since $x \propto (T_B/T)^{1-g}$, the terms $f_{2n}^{(\mathrm{BSG})} x^{2n}$ and $f_{2n} x^{2n}$ are proportional to T_B when $g = 1 - 1/(2n)$. Therefore, there must be a simple pole in $f_{2n} x^{2n}$ at $g = 1 - 1/(2n)$, with residue $r_{2n} \equiv T_B/(2\pi n^2 T)$. The pole in $f_{2n}^{(\mathrm{BSG})} x^{2n}$ has residue $(-1)^{n+1} r_{2n}/2$. By the same argument, the free energy coefficients in the spin-$j/2$ Kondo model when j is odd each has a single pole at $g = 1 - 1/(2n)$ with residues r_{2n}. The free energy coefficients for the integer-spin Kondo model have no pole at these values.

We can see these poles explicitly by studying the series expansion (2.6) for the I_{2n} in the boundary sine-Gordon model. Initially the series looks useless, because it diverges for $g \geqslant 1/2$, and we only know how to resume it for I_2. However, a first interesting observation is that the series expressions for the $f_{2n}^{(\mathrm{BSG})}$ converge even where those for the individual I_{2n} do not. This is because some of the divergences in the Coulomb integrals are cancelled when taking the connected part. More precisely, we define the truncated series $I_{2n}(\Lambda)$ as the expression (2.6) with all $m_i \leqslant \Lambda$ and

$$f_4^{(\mathrm{BSG})}(\Lambda) \equiv \frac{[I_2(\Lambda)]^2}{2} - I_4(\Lambda)$$

Then, using the previously obtained recurrence relation (2.7), one finds that

$$f_4^{(\mathrm{BSG})}(\Lambda) - f_4^{(\mathrm{BSG})}(\Lambda - 1) \simeq \frac{2g(1-2g)+1}{2(1-2g)} \Gamma(g)^{-4} \Lambda^{4g-4} \tag{3.1}$$

for Λ large. This expression converges as $\Lambda \to \infty$ for $g < 3/4$. Moreover, the pole at $g = 3/4$ is clearly identified, and its residue can easily be computed, because the divergence is proportional to that of the zeta function. One confirms the earlier result that near $g = 3/4$

$$f_4^{(\mathrm{BSG})} x^4 \approx -\frac{1}{16\pi(g-3/4)} \left(\frac{T_B}{T}\right)^{4(1-g)}$$

where we used the relation (2.20) to relate x and T_B.

Since (3.1) tells us explicitly how the series diverges, the continuation around the pole can be constructed by adding and subtracting a zeta function. More precisely, we define the continuation to be

$$f_4^{(\mathrm{BSG})}(\infty) = f_4^{(\mathrm{BSG})\mathrm{reg}}(\infty) + \frac{2g(1-2g)+1}{2(1-2g)\,\Gamma(g)^4} \zeta(4-4g) \tag{3.2}$$

where

$$f_4^{(\mathrm{BSG})\mathrm{reg}}(\Lambda) \equiv f_4^{(\mathrm{BSG})}(\Lambda) - \frac{2g(1-2g)+1}{2(1-2g)\,\Gamma(g)^4} \sum_{n=1}^{\Lambda} n^{4g-4}$$

The "regular" part $f_4^{(\mathrm{BSG})\mathrm{reg}}$ is $f_4^{(\mathrm{BSG})}$ with the diverging part of the sum subtracted off. This series converges when $\Lambda \to \infty$ for $g < 1$. One can extend this result past $g = 1$ in the same manner.

It should be possible to find all the f_{2n} for $g<1$ in this manner. One first writes the higher f_{2n}'s in terms of the I_{2n} using the relation

$$f_{2n}^{(\mathrm{BSG})} = \sum_{\mathbf{m}} \frac{(-1)^{l(\mathbf{m})-1}(l(\mathbf{m})-1)!}{\prod_j \lambda_j!} I_{2(n)} \tag{3.3}$$

where $\mathbf{m} = \{m_1, m_2,..., m_{l(\mathbf{m})}\}$ is a partition of an integer so that $\sum m_i = n$, $I_{2(n)} \equiv I_{2m_1} I_{2m_2} \cdots$, and λ_j is the multiplicity of the integer j in $\mathbf{m}$. We have checked that $f_6^{(\mathrm{BSG})}(\Lambda) - f_6^{(\mathrm{BSG})}(\Lambda-1) \to C_6(g)\, \Lambda^{6g-6}$ and $f_8^{(\mathrm{BSG})}(\Lambda) - f_8^{(\mathrm{BSG})}(\Lambda-1) \to C_8(g)\, \Lambda^{8g-8}$ when Λ is large, with $C_6(g)$, $C_8(g)$ known expressions. Thus poles in $f_{2n}^{(\mathrm{BSG})}$ appear at $g = 1 - 1/2n$ for $n = 1, 2, 3, 4$ with the appropriate residue. We can then apply the same zeta-function method and regularize the sums to go all the way to $g = 1$.

We have checked that the numerical values agree very well with the Bethe ansatz results. This takes some effort because finding the numbers from the Bethe ansatz requires that we numerically solve the integral equations, and then numerically fit the results to a power series. Moreover, at

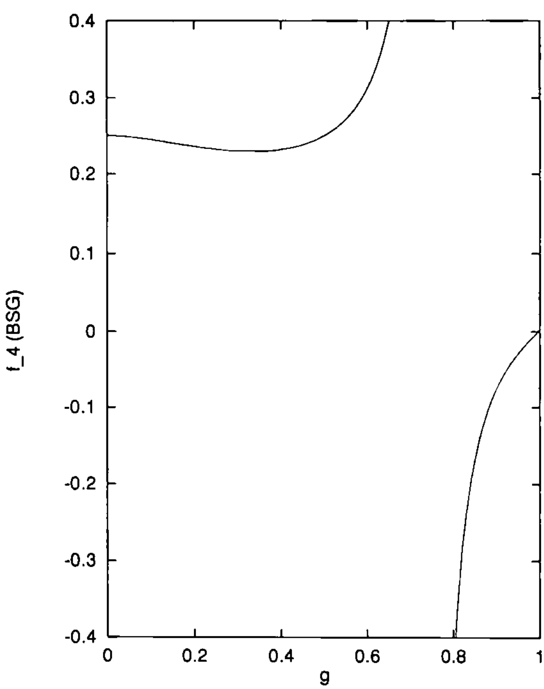


Fig. 2. The free-energy coefficient $f_4^{(\mathrm{BSG})}$ as a function of g. The pole is at $g = 3/4$.

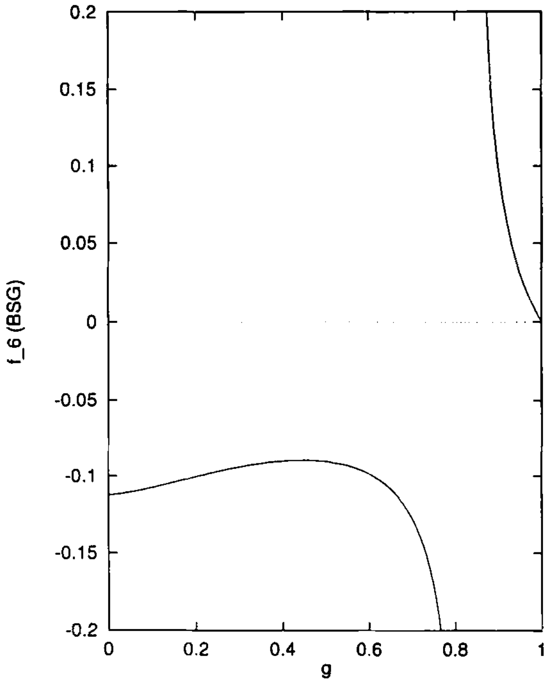


Fig. 3. The free-energy coefficient $f_6^{(\mathrm{BSG})}$ as a function of g. The pole is at $g = 5/6$.

$g = 1$, we have $Z_{\mathrm{BSG}}(x) = 1$. We plot the results for $f_4^{(\mathrm{BSG})}$ and $f_6^{(\mathrm{BSG})}$ in Figs. 2 and 3. We clearly see the pole in the data, and that $f_4^{(\mathrm{BSG})}$ and $f_6^{(\mathrm{BSG})}$ do indeed go to zero as $g \to 1$. Another interesting consequence is that it allows a very simple approximation formula for the f_{2n} or $f_{2n}^{(\mathrm{BSG})}$ for g near 1. We approximate the function by its pole plus a constant piece. For example, if we define the constant piece by requiring that $f_{2n}^{(\mathrm{BSG})}(g = 1) = 0$, we have

$$f_{2n}^{(\mathrm{BSG})}(g) \approx \frac{\Gamma(n - 1/2)}{2\sqrt{\pi}\, n^2 \Gamma(n)\, \Gamma(1 - 1/2n)^{2n}} \left[2n + \frac{1}{1 - g - (1/2n)} \right]$$

The presence of these poles has interesting physical consequences. They indicate that the free energy at $g = 1 - 1/(2n)$ cannot be expanded as a power series, but has an additional logarithmic term. The TBA free energy does not have a divergence, because the pole is subtracted off, as in (2.22). However, there is a leftover piece:

$$\lim_{g\to 1-1/(2n)}\left[T_B\tan\left(\frac{\pi g}{2(1-g)}\right)+\frac{Tr_{2n}}{g-1+1/(2n)}\left(\frac{T_B}{T}\right)^{2n(1-g)-1}\right]$$
$$=-2nTr_{2n}\ln\left(\frac{T_B}{T}\right)+\cdots$$

Thus the free energy contains a logarithmic correction, arising from proper regularization of the divergence. Such terms are not unusual; for example, they occur in the bulk free energy of the 2D Ising model and its (even) multicritical generalizations.[27] As we will see in the next subsection, the existence of the log term also follows from the detailed TBA analysis. The fact that the free energy defined by analytic continuation has a simple pole at r_{2n} does not mean that the physical free energy—obtained with a cutoff regularization—diverges, but rather indicates that it has a logarithmic dependence on the cutoff at that point.[26]

These results can be checked for several special values of g. In the Toulouse limit $g=1/2$, the model is equivalent to a free fermion in a boundary magnetic field. As discussed in ref. 2, for example, the free energy for spin-1/2 Kondo is

$$F_2=-2T\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\frac{1}{\cosh(\theta-\ln T_B/T)}\ln(1+e^{-e^{\theta}})$$
$$=2T\ln\Gamma\left(\frac{T_B}{\pi T}\right)-2T\ln\Gamma\left(\frac{T_B}{2\pi T}\right)+\frac{T_B}{\pi}\left(1-2\ln 2-\ln\frac{T_B}{2\pi T}\right)-T\ln 2$$

Using the gamma function identity $\Gamma(2a)=\Gamma(a)\,\Gamma(a+1/2)\,2^{2a-1}/\sqrt{\pi}$ and the expansion

$$\ln\Gamma\left(a+\frac{1}{2}\right)=\ln\left(\frac{1}{2}\right)+\sum_{n=1}^{\infty}\psi^{(n-1)}\left(\frac{1}{2}\right)\frac{a^n}{n!}$$

where $\psi^{(m)}(x)$ is the mth derivative of the digamma function $\psi(x)\equiv\Gamma'(x)/\Gamma(x)$, one finds

$$F_2=-T\ln 2+\frac{T_B}{\pi}\left(1-\ln\frac{T_B}{2\pi T}\right)+2T\sum_{n=1}^{\infty}\psi^{(n-1)}\left(\frac{1}{2}\right)\frac{1}{n!}\left(\frac{T_B}{2\pi T}\right)^n \tag{3.4}$$

Thus we see explicitly the log term at $g=1/2$, with coefficient $r_2=1/2\pi$ as derived above. Moreover, we see that all the f_{2n} for $n>1$ are finite at $g=1/2$. This means, for example, that there is a double pole in Q_4 at $g=1/2$ in order for $f_4=-Q_4/2+Q_2^2/8$ to remain finite. In fact, this means that there is an nth-order pole in Q_{2n} at $g = 1/2$ and that it is analytic in the neighborhood (and in fact all the way to $g = 3/4$). Moreover, at $g = 1/2$, $F_{\mathrm{BSG}} = F_1/2$, and we have checked numerically that the values for $f_{2n}^{(\mathrm{BSG})}$ from (3.4) are obtained by taking the limit of the series expression as $g \to 1/2$, (2.6).

A final comment is in order. At the isotropic point $g = 1$, the BSG model is trivial with these boundary conditions, so $Z_{\mathrm{BSG}}(x) = 1$ and $I_{2n} = 0$. Notice that this follows easily from the relation (2.15). However, the Kondo problem is not trivial at $g = 1$ (this is the value of most physical interest), but the power series expression (proportional to T_B^{1-g}) obviously requires modification. The fact that the exponent is vanishing is an obvious hint that there are log terms at every order, and indeed this is seen in the TBA solution.[1,2] Notice that the shift between the TBA and the power series has an essential singularity as $g \to 1$, so if subtracted appropriately from the power series as $g \to 1$, the result may be finite and give the series with log terms at $g = 1$. We have not yet succeeded in carrying out this analysis.

3.2. The TBA in the Repulsive Regime

The TBA equations for the Kondo problem in the repulsive regime were derived in ref. 2. For technical simplicity, we consider only $g = 1 - 1/s$, $s \geq 2$ an integer. The equations are very similar to those in the attractive regime:

$$\varepsilon_j = \delta_{j1} e^{\theta} - \sum_k N_{jk} \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln(1 + e^{-\varepsilon_k(\theta')}) \tag{3.5}$$

where the incidence matrix N_{jk} is as in Section 2.3 with t replaced by s. The Kondo free energy is

$$\begin{aligned} F_j &= -T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \ln T_B/T)} \ln(1 + e^{-\varepsilon_j}) \\ F_{s-1}^{\cdot} &= -2T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \ln T_B/T)} \ln(1 + e^{-\varepsilon_{s-1}}) \end{aligned} \tag{3.6}$$

where $\varepsilon_+ = \varepsilon_- \equiv \varepsilon_{s-1}$. Even though the BSG problem is integrable in this regime, applying the TBA is difficult technically since now both the bulk and the boundary scattering matrices are not diagonal. We will use analytic continuation again to provide the BSG free energy.

Defining this time

$$Y_j(\alpha) = e^{-\varepsilon_j(\alpha)}$$

we see right away that for $j = 2, ..., s-2$,

$$\begin{aligned} Y_1 &= e^{-F_2/T} e^{-e^{\alpha}} \\ Y_j &= e^{-F_{j+1}/T} e^{-F_{j-1}/T} \\ Y_{s-1} &= e^{-F_{s-2}/T} \end{aligned} \tag{3.7}$$

analogous to (2.26). Arguments identical to those in the attractive case require that $Y(\alpha + is\pi) = Y(\alpha)$.[25] Thus Y can be expressed as an analytic power series in $x^2 \propto e^{2\alpha/s} = (T_B/T)^{2(1-g)}$ as before. Therefore, (3.7) indicates that for j even, $F_j + e^{\alpha}$ is a power series in x^2. When j is odd *and* s is odd, F_j is a power series in x^2 as well, but for s even and j odd, any other term is allowed as well. In fact there is a log term, as discussed in the last subsection.

We can find the log terms at $g = 1 - 1/(2n)$ (i.e., s even) directly from the TBA, by plugging the power series expansion for Y into (3.6). For example, for $s = 4$ $(g = 3/4)$, $Y_1(\alpha) = 3 + ae^{\alpha/2} + be^{\alpha}$. Then, we see that

$$\frac{F_1}{T} + \ln 2 - f_2 x^2 = -\int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \ln T_B/T)} \left\{ \ln[1 + Y_1(\theta)] - \ln 4 - \frac{a}{4} e^{\theta/2} \right\}$$

For T_B/T small, this is approximately

$$\begin{aligned} &-\int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \ln T_B/T)} \ln \left(\frac{4 + ae^{\theta/2} + (b - a^2/8)\, e^{\theta}}{4 + ae^{\theta/2}} \right) \\ &\approx -\int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \ln T_B/T)} \frac{(b - a^2/8)\, e^{\theta}}{4 + ae^{\theta/2}} \\ &= \frac{4T_B}{2\pi T} \left(b - \frac{a^2}{8} \right) \int_0^{\infty} du \, \frac{u^3}{1 + u^4} \frac{1}{4 + a(T_B/T)^{1/2}\, u} \\ &\approx \frac{b - a^2/8}{4\pi} \frac{T_B}{T} \ln \frac{T_B}{T} \end{aligned}$$

One can in fact verify using functional relations analogous to those above and in ref. 11 that $b - a^2/8 = -2$, so the coefficient is indeed $-4r_4 = -T_B/(2\pi T)$, as shown in the previous subsection.

We can now derive the analogs of the fusion relations (2.14). we define

$$\tilde{Z}_j(x) = e^{-F_j(\alpha)/T} \qquad j \text{ odd}$$
$$\tilde{Z}_j(x) = e^{-F_j(\alpha)/T} e^{-e^{\alpha}} \qquad j \text{ even}$$

Using (2.27), we have

$$\tilde{Z}_j(-q)^{1/2}x)\,\tilde{Z}_j((-q)^{-1/2}x) = 1 + Y_j = 1 + \tilde{Z}_{j-1}(x)\,\tilde{Z}_{j+1}(x) \qquad (3.8)$$

analogous to (2.28) and (2.29) in the attractive regime. The crucial difference is that q has been replaced by $-q^{-1}$. The $\tilde{Z}_j(x)$ are analytic functions of x^2 for j odd, but for j even they include the log term, so implicit in this equation is the prescription $-i\pi < \text{Im} \ln y < i\pi$. Because the analytic continuation of Z_j should still satisfy the fusion relation (2.29), not all of the $\tilde{Z}_j$ can be the analytic continuation of the Z_j from the attractive regime to the repulsive regime. However, notice that if we make the identification

$$\ln \tilde{Z}_j(x) = \ln Z_j(x) + \frac{T_B}{T}\frac{\sin j\pi(s-1)/2}{\cos \pi(s-1)/2}, \qquad j \text{ odd}$$

$$\ln \tilde{Z}_j(x) = \ln Z_j(ix) + \frac{T_B}{T}\frac{\sin j\pi(s-1)/2}{\cos \pi(s-1)/2}, \qquad j \text{ even}$$

then the Z_j satisfy the fusion relations (2.29). The shift as before cancels the pole in $\ln Z_j$ for j odd. Since Z_j with j even is a power series in x^2, the effect of the argument ix is to flip the sign of every other term. This is merely a matter of convention. With our choice $q = e^{i\pi g}$, $q = 1$ for the classical limit $g \to 0$, while $q = -1$ at the $SU(2)$ point $g = 1$. Representations of $SU(2)_q$ and $SU(2)_{-q^{-1}}$ are identical for j odd, but they differ by a factor of i in the matrix elements of $S_{\pm}$ for j even. The coupling renormalization for j even would disappear if we chose to change the quantum group conventions.

We can now find Z_{BSG} by analytically continuing the functional relation (2.15). Since this relation involves only q and the functions are series in x^2, we can replace q by $-q^{-1}$. With this replacement, all functional relations derived in in Section 2.3 apply to the repulsive regime with t replaced by s. In particular, we showed that $2F_{\text{BSG}}(\xi x) = F_{t-1}(x)$ implies (2.15). Since the relation (2.15) for $g \neq 1$ determines all of the BSG coefficients I_{2n} uniquely in terms of the spin-1/2 ones Q_{2n}, given a Z_1, it determines Z_{BSG} uniquely. Therefore, we can reverse the argument in Section 2.3, and say that given (2.15), we must have

$$2F_{\text{BSG}}(\xi x) = F_{s-1}(x) \qquad (3.9)$$

where $\xi = i/(q - q^{-1})$ as before. This result (without specifying the ξ) was conjectured in ref. 28. We emphasize that this relation is only true for s integer. The relation (2.15) of course is true for any value of g. However, the exact result at integer s is very useful, allowing us, for example, to find the conductance in the BSG model exactly at these values, without any analytic continuation.

We have checked the result (3.9) numerically at $s = 3$ (again by comparing analytic continuation results to TBA ones) and find good agreement. The relation (2.20) still holds in the repulsive regime (the derivation of ref. 12 holds for all g), and we confirm also the value of ξ.

An analytic check can be done at the special point $g = 3/4$, where the boundary sine-Gordon model is equivalent to the Toulouse limit of the four-channel Kondo model.[29] This model can be solved exactly,[30] and the above results reproduce these.

4. NONZERO VOLTAGE

We now extend the results of Section 2 to allow for nonzero voltage in the BSG problem and nonzero magnetic field in the Kondo model. For the Kondo problem, the TBA analysis is easily extended to nonzero magnetic field.[1,2] The analysis is straightforward because the magnetic field couples to a conserved charge, the z component of the spin. Since the charge commutes with the Hamiltonian, the same diagonalization applies even with a magnetic field. However, in the BSG problem the voltage violates the charge conservation. Indeed, this is responsible for the charge tunneling in the Luttinger liquid with an impurity. In the presence of a voltage in the BSG model, current flows. This is a problem out of equilibrium, where transport properties can be computed using a kinetic equation.[7,11,12,18] However, the partition function can be formally extended to nonzero voltage. As a byproduct, we will find some more information about the transport properties. In particular, we conjecture relations for the conductance good for all values of g, generalizing the results of g, generalizing the results of refs. 7, 11, and 12.

The partition functions can be expanded in powers of x as before:

$$
\begin{aligned}
Z_1(x, p) &= 2 \cos p\pi + \sum_{n=1}^{\infty} x^{2n} Q_{2n}(p) \\
Z_{\mathrm{BSG}}(x, p) &= 1 + \sum_{n=1}^{\infty} x^{2n} I_{2n}(p)
\end{aligned}
\tag{4.1}
$$

where

$$Q_n(p)=\int_0^{2\pi} du_1 \cdots \int_0^{u_n} du'_n$$

$$\times \mathscr{I}_{2n}(\{u_i\},\{u'_i\})\, 2\cos p\left(\pi+\sum_i (u'_i-u_i)\right)$$

$$I_{2n}(p)=\frac{1}{(n!)^2}\int_0^{2\pi} du_1 \cdots \int_0^{2\pi} du'_n \times \mathscr{I}_{2n}(\{u_i\},\{u'_i\}) \exp\left(ip\sum_i (u_i-u'_i)\right)$$

where $p=igV/2\pi T$ is integer. In ref. 11 exact series expressions for the $I_{2n}(p)$ were found for integer p:

$$I_{2n}(p)=\frac{1}{\Gamma(g)^{2n}}\sum_{\mathbf{m}}\prod_{i=1}^{n}\frac{\Gamma[m_i+g(n-i+1)]\,\Gamma[p+m_i+g(n-i+1)]}{\Gamma[m_i+1+g(n-i)]\,\Gamma[p+m_i+1+g(n-i)]} \tag{4.2}$$

where $\mathbf{m}$ is defined as in (2.6). As before, this series converges only for $g<1/2$.

These results apply only to p integer. We now use (4.1), (4.2) to *define* Z_1 and Z_{BSG} for complex p. We conjecture this is the unique analytic continuation. This, presumably, could be proven using information on the $p\to\infty$ behaviour. Indeed, we know that $I_{2n}(p)/T^{2n(1-g)}$ has a limit as $p\to i\infty$ (where $T\to 0$), and we assume this applies at real p as well. This assumption has been checked with TBA results below. For instance, the series that defines I_2 for complex p can be summed

$$I_2(p)=\sum_{m_1=0}^{\infty}\frac{\Gamma(g+m_1)\,\Gamma(g+m_1+p)}{\Gamma^2(g)\,\Gamma(1+m_1)\,\Gamma(1+m_1+p)}$$

$$=\frac{\sin\pi g\Gamma(1-2g)}{\sin\pi(g+p)\,\Gamma(1-g+p)\,\Gamma(1-g-p)} \tag{4.3}$$

One can indeed check that this goes like $p^{2(g-1)}$ as $p\to i\infty$ to reproduce the zero-temperature coefficient of ref. 12, giving support to our assumption. Notice that the continuation of $I_{2n}(p)$ is not even in p, and that $Z_{\mathrm{BSG}}(p)$ is real only for p integer. However, we will see that observable quantities like the conductance are given in terms of $Z_{\mathrm{BSG}}(p)$.

The analysis of Section 2.2 can be repeated for p integer and nonzero. The fusion relations (2.14) apply without modification for spin-$j/2$ representations. The k-dimensional cyclic representation when $q^k=\pm 1$

also obeys the same relation, but to find the BSG free energy, there is a subtlety. We first note that when $p \neq 0$ we have

$$Z_\delta(x, p) \approx \left(\sum_{j=0}^{k-1} e^{i\pi jp} \right) Z_{\mathrm{BSG}} \left(\frac{Cq^{1/2}}{q - q^{-1}} x, p \right), \qquad q^\delta = C \gg 1 \tag{4.4}$$

when the fundamental set of S_z values is taken to be $0, \dots, k-1$ (recall that periodic representations are invariant under overall shifts of S_z). When one fuses such a spin-δ representation with a spin-1/2 representation, the last relation in (2.14) still applies, but the new cyclic representations have shifted fundamental sets, $1, \dots, k$ and $-1, \dots, k-2$, respectively. Therefore the relation between Z_1 and Z_{BSG} is slightly modified:

$$Z_1[(q - q^{-1}) x, p] = \frac{e^{i\pi p} Z_{\mathrm{BSG}}(qx, p) + e^{-i\pi p} Z_{\mathrm{BSG}}(q^{-1}x, p)}{Z_{\mathrm{BSG}}(x, p)} \tag{4.5}$$

We assume this relation still holds for p real, where both sides are defined through (4.1). Plugging in the power series expansions into (4.5) gives the $I_{2n}(p)$ in terms of the $Q_{2n}(p)$. For example,

$$I_2(p) = \frac{\sin g\pi}{\sin \pi(g+p)} Q_2(p)$$

in agreement with (4.3). At next order, we find similarly

$$\begin{aligned} 4 \sin^3(\pi g)\, Q_4(p) = {} & -2 \cos \pi g \sin[\pi(2g+p)]\, I_4(p) \\ & + \sin[\pi(g+p)]\, I_2(p)^2 \end{aligned} \tag{4.6}$$

generalizing the $p = 0$ relations in ref. 15. This relation (4.5) has been checked, again by numerical determination of the TBA results for the Kondo model at nonzero magnetic field.

The functional relation (4.5) extended to complex p implies even more than the Kondo partition function. For example, we know on physical grounds [and from the definition (4.1)] that the Kondo partition function is even in p. Combining this with (4.5) then yields a nontrivial functional relation for Z_{BSG}:

$$\begin{aligned} & e^{i\pi p}[Z_{\mathrm{BSG}}(qx, p)\, Z_{\mathrm{BSG}}(x, -p) - Z_{\mathrm{BSG}}(q^{-1}x, -p)\, Z_{\mathrm{BSG}}(x, p)] \\ & \quad = e^{-i\pi p}[Z_{\mathrm{BSG}}(qx, -p)\, Z_{\mathrm{BSG}}(x, p) - Z_{\mathrm{BSG}}(q^{-1}x, p)\, Z_{\mathrm{BSG}}(x, -p)] \end{aligned} \tag{4.7}$$

Plugging in the perturbative expansion gives the values of the coefficients $I_{2n}(-p)$ in term of the $I_{2n}(p)$:

$$I_2(p)\sin[(g+p)] = I_2(-p)\sin[\pi(g-p)]$$

$$I_4(p)\sin[(2g+p)] - I_4(-p)\sin[\pi(2g-p)] = -\sin(\pi p)\, O_2(p)\, I_2(-p)$$

for example.

In ref. 11 we made and checked a conjecture which related the linear-response conductance directly to the partition function. Our conjecture at general $\mu = V/2T$ and g is

$$G\left(x, \frac{V}{2T}\right) = g - \frac{ig\pi x}{2} \frac{\partial}{\partial(V/2T)} \frac{\partial}{\partial x} \ln\left(\frac{Z_{\mathrm{BSG}}(x, ig\mu/\pi)}{Z_{\mathrm{BSG}}(x, -ig\mu/\pi)}\right) \tag{4.8}$$

where we stress again that Z_{BSG} is defined through (4.1), (4.2). We have checked this generalized expression numerically as well, by comparing it to the conductance from the Boltzmann equations in refs. 7 and 12 and below.

As at $V=0$, much is learned by comparing the perturbative and TBA analyses. The TBA equations (2.16) at $g=1/t$ are modified slightly in the presence of a finite voltage, yielding[1,2]

$$\varepsilon_j = \sum_k N_{jk} s_{t-1} * \ln(1 + e^{\mu_k} e^{\varepsilon_k}) \tag{4.9}$$

where the incidence diagram is as in Section 2. A magnetic field corresponds to the chemical potentials $\mu_\pm = \pm\mu$, $\mu_k = 0$ otherwise, and $\mu \equiv V/2T$. The two end nodes of the incidence diagram now play a different role, but we still have $\varepsilon_+ = \varepsilon_- \equiv \varepsilon_{t-1}$. The relation

$$\ln Z_j(x, \mu) = s_{t-1} * \ln(1 + e^{\varepsilon_j}), \qquad j = 1, \ldots, t-2$$

still holds. One also has

$$\ln Z_{t-1}(x, \mu) = s_{t-1} * [\ln(1 + e^{\mu} e^{\varepsilon_{t-1}}) + \ln(1 + e^{-\mu} e^{\varepsilon_{t-1}})] \tag{4.10}$$

We then *define*

$$\ln Z_\pm(x, \mu) = -\frac{1}{2} \ln\left[1 + e^{\pm\mu} \frac{\sinh[(t-1)\mu/t]}{\sinh(\mu/t)}\right] + s_{t-1} * \ln(1 + e^{\pm\mu} e^{\varepsilon_{t-1}}) \tag{4.11}$$

Using the same identity (2.27) as for $\mu=0$ yields

$$Z_\pm(q^{1/2}x,\mu)\,Z_\pm(q^{-1/2}x,\mu)=\left[1+e^{\pm\mu}\frac{\sinh[(t-1)\,\mu/t]}{\sinh(\mu/t)}\right]^{-1}(1+Y_\pm)$$
$$Z_{t-1}=\frac{\sinh\mu}{\sinh(\mu/t)}Z_+Z_- \tag{4.12}$$

where

$$Y_\pm=e^{\pm\mu}e^{\varepsilon_{t-1}}$$

Similarly, relation (2.31) becomes

$$Z_t(x,\mu)=Z_{t-2}(x,\mu)+2\cosh\mu, \tag{4.13}$$

since the two additional states in the spin-t representation have third component of the spin equal to $\pm t$. Fusion identities like (2.29) carry over to the case of finite voltage. Following the same arguments as for $\mu=0$, one finds then

$$Z_1[(q-q^{-1})\,x,\mu]=e^{-\mu/t}\frac{Z_+(qx,\mu)}{Z_-(x,\mu)}+e^{\mu/t}\frac{Z_-(q^{-1}x,\mu)}{Z_+(x,\mu)} \tag{4.14}$$

where we remind the reader that $\mu=V/2T$.

The functions $Z_\pm$ are not obviously related to any Kondo-type integrals. Since there are technical obstacles to directly calculating the boundary sine-Gordon partition function in the presence of a voltage, we proceed using the algebraic approach. Comparing (4.14) and (4.5) suggests the functional relations

$$\frac{Z_{\mathrm{BSG}}(q^{1/2}x,i\mu/\pi t)}{Z_{\mathrm{BSG}}(q^{-1/2}x,i\mu/\pi t)}=\frac{Z_+(q^{1/2}x,\mu)}{Z_-(q^{-1/2}x,\mu)} \tag{4.15}$$

together with

$$Z_{\mathrm{BSG}}\left(x,\frac{i\mu}{\pi t}\right)Z_{\mathrm{BSG}}\left(x,-\frac{i\mu}{\pi t}\right)=\frac{\sinh(\mu/t)}{\sinh\mu}Z_{t-1}(x,\mu) \tag{4.16}$$

Here we traded the p variable of (4.1) for the μ variable. It is very likely that (4.16) has an algebraic origin. This is because the tensor product of two cyclic representations decomposes on pairs of (generally) indecomposable representations, which in turn are related with the spin-$(t-1)$ representation of vanishing q dimension. However, we have failed in finding a complete algebraic proof of (4.16), but we have checked it thoroughly by using the series expressions above for the Z_{BSG} and the numerical TBA results for Z_{t-1}.

We can check that (4.15) is consistent with the conjectured relation (4.8) between the partition function and the conductance. Using the TBA and a kinetic equation, the conductance at integer $t = 1/g$ is[7, 12]

$$G\left(x, \frac{V}{2T}\right) = \frac{T(t-1)}{2} \frac{d}{dV} \int \frac{d\theta}{\cosh^2(t-1)(\theta-\alpha)} \times \ln\left(\frac{1+e^{V/2T}e^{-\varepsilon_{t-1}}}{1+e^{-V/2T}e^{-\varepsilon_{t-1}}}\right) \tag{4.17}$$

Using the identity

$$\lim_{x\to 0}\left[\frac{1}{\cosh^2(\theta+i\pi/2-x)} - \frac{1}{\cosh^2(\theta-i\pi/2+x)}\right] = -2i\pi\delta'(\theta) \tag{4.18}$$

it follows that

$$G\left(q^{1/2}x, \frac{V}{2T}\right) - G\left(q^{-1/2}x, \frac{V}{2T}\right) = -\frac{i\pi x}{2t}\frac{\partial}{\partial x}\frac{\partial}{\partial(V/2T)} \ln\left(\frac{1+e^{V/2T}e^{\varepsilon_{t-1}}}{1+e^{-V/2T}e^{\varepsilon_{t-1}}}\right) \tag{4.19}$$

This allows a powerful check on the conjectures (4.8) and (4.15), because it also follows from substituting (4.15) into (4.8) and using the definition of $Z_\pm$, (4.11). It would be nice to reverse the order of the proof and show that (4.19) (known to be true from the TBA) implies (4.8) and (4.15). This cannot be done by substituting the perturbative expansion because the relation (4.19) does not determine all of them uniquely; the order-x^{jt} term vanishes on the left-hand side for any integer j. However, it is conceivable that by exploiting additional analyticity information it could be proven along the lines of ref. 31.

In the linear-response limit $V \to 0$, we can recover another functional relation from ref. 11. In this limit we can ignore the V dependence of Y_{t-1} because it is a function of V^2. Using (2.30), we recover

$$G(q^{1/2}x, 0) - G(q^{-1/2}x, 0) = i\pi g^2 x \frac{\partial}{\partial x}\frac{1}{Z_{BSG}(q^{1/2}x, 0)\, Z_{BSG}(q^{-1/2}x, 0)} \tag{4.20}$$

This formula is nice because it no longer has any reference to the TBA quantities ε, so we expect it to hold for all g.

We have a formula (4.16) which relates the product of $Z_{\mathrm{BSG}}(V)$ and $Z_{\mathrm{BSG}}(-V)$ to TBA quantities, and a formula (4.8) which relates their ratio to TBA quantities by using (4.17). Therefore, we can infer a complete expression for $Z_{\mathrm{BSG}}(x, \mu)$ alone in terms of the TBA quantities:

$$\begin{aligned}
\ln Z_{\mathrm{BSG}}&\left(x, \frac{iV}{2\pi tT}\right)\\
&=\frac{t-1}{2\pi}\int d\theta\left\{\frac{e^{(t-1)(\theta-\alpha)}-i}{1+e^{2(t-1)(\theta-\alpha)}}\ln\left(\frac{1+e^{\varepsilon_{t-1}}e^{-V/2T}}{1+e^{\varepsilon_{t-1}(-\infty)}e^{-V/2T}}\right)\right.\\
&\quad\left.+\frac{e^{(t-1)(\theta-\alpha)}+i}{1+e^{2(t-1)(\theta-\alpha)}}\ln\left(\frac{1+e^{\varepsilon_{t-1}}e^{V/2T}}{1+e^{\varepsilon_{t-1}(-\infty)}e^{V/2T}}\right)\right\}
\end{aligned} \tag{4.21}$$

where

$$e^{\varepsilon_{t-1}(-\infty)}=\frac{\sinh[(t-1)\mu/t]}{\sinh(\mu/t)}$$

and $\alpha=\ln(T_B/T)$ and $\mu=V/2T$ as always; x is related to α via (2.20). It should be possible to derive this directly from the TBA, but there are some technical obstacles.

Extending this analysis to the repulsive regime is more difficult. The reason is that for integer s in $g=1-1/s$, the value of S_z at the top and bottom states of the spin-s representation still is $\pm s$; *not* $s/(s-1)$ as would be needed to carry over the algebra of the attractive regime straightforwardly. This issue is related to the q versus $-q^{-1}$ problem we had to address in Section 3.2. We find after some manipulation that

$$\frac{Z_{\mathrm{BSG}}(q^{1/2}x, i\mu/\pi s)}{Z_{\mathrm{BSG}}(q^{-1/2}x, i\mu/\pi s)}=\frac{Z_+(q^{1/2}x,\mu)}{Z_-(q^{-1/2}x,\mu)} \tag{4.22}$$

as above, but with, however, the new correspondence $\mu=(s-1)\,V/2T$.

We can also propose a formula for the conductance in the repulsive regime. By using the conjectures (4.16) and (4.8) and the identity (2.27), we now deduce

$$\begin{aligned}
G&\left(q^{1/2}x, \frac{V}{2T}\right)-G\left(q^{-1/2}x, \frac{V}{2T}\right)\\
&=-\frac{i\pi x}{2}\left(1-\frac{1}{s}\right)\frac{\partial}{\partial x}\frac{\partial}{\partial(V/2T)}\ln\left[\frac{1+e^{(s-1)V/2T}e^{-\varepsilon_{s-1}}}{1+e^{-(s-1)V/2T}e^{-\varepsilon_{s-1}}}\right]
\end{aligned}$$

from which we obtain

$$G\left(x, \frac{V}{2T}\right) = \frac{s-1}{2} \int d\theta \frac{1}{\cosh^2(\theta - \ln T_B/T)}$$
$$\times \frac{d}{d(V/T)} \left\{ \ln \left[\frac{1 + e^{(s-1)V/2T} e^{-\varepsilon_{s-1}(\theta)}}{1 + e^{-(s-1)V/2T} e^{-\varepsilon_{s-1}(\theta)}} \right] - \ln \left[\frac{1 + e^{(s-1)V/2T} e^{-\varepsilon_{s-1}(\infty)}}{1 + e^{-(s-1)V/2T} e^{-\varepsilon_{s-1}(\infty)}} \right] \right\} \tag{4.23}$$

By generalizing the result of Eq. (2.21) to nonzero V, one can check that these formulas give the correct limit $G(0, V/2T) = (1 - 1/s)$. In the limit of vanishing voltage (linear response), one finds

$$G(x, 0) = \frac{(s-1)^2}{2} \int d\theta \frac{1}{\cosh^2(\theta - \ln T_B/T)} \left(\frac{1}{1 + e^{\varepsilon_{s-1}(\theta)}} - \frac{1}{1 + e^{\varepsilon_{s-1}(\infty)}} \right) \tag{4.24}$$

This expression has been compared to real-time Monte Carlo simulations in ref. 32; the agreement is good. At $T = 0$, it agrees with the expression derived in ref. 12. Using the identity (4.18) with (4.24) and using the TBA expression for the free energy from Section 3.2 yields the functional relation (4.20) in the repulsive regime, lending support to the conjecture that (4.20) holds for all g.

5. CONCLUSION

It should be possible to use the identification of the boundary sine-Gordon model with a Kondo-type problem more completely than we have done. One way of doing so would be to write and solve the Bethe ansatz equations for an integrable system made of spins 1/2 and a cyclic impurity. Since (1.3) conserves the charge, unlike (1.2), this would allow one to handle directly the BSG model with a voltage, avoiding the lengthy series of functional identities of Section 4.

Another interesting direction is to try to continue the perturbative Anderson–Yuval coefficients past $g = 1$ into the irrelevant regime. This can be done by our zeta-function trick of Section 3: use the explicit series expression to find out how the series is diverging, and subtract and add the appropriate Zeta function. This is straightforward but tedious, so we have not completed this program. For example, the duality[3] $g \to 1/g$ should be explicitly observable. So far this duality has been established only at vanishing temperature.[11] Observe, however, that such a way of handling irrelevant operators does not involve any cutoff, and will not describe the nonuniversal physics depending on the cutoff (e.g., the dissipative quantum mechanics in ref. 33). However, our result does provide the interesting prospect of a well-controlled irrelevant perturbation theory, defined by an analog of dimensional regularization.

APPENDIX. DERIVING THE *S* MATRIX FOR BOUNDARY SINE-GORDON

The boundary S matrix of the boundary sine-Gordon model was found in ref. 21 by analyzing the most general solution of the boundary Yang–Baxter equation. Here we show that this form also follows from the identification of the BSG model with a cyclic-spin anisotropic Kondo model.

For the ordinary spin-$j/2$ Kondo model, the matrix for a particle scattering off the impurity is easy to obtain. Up to an overall proportionality factor which follows from crossing and unitarity, it is simply the standard Yang–Baxter solution for a spin $(j-1)/2$ and a spin $1/2$, and a renormalized quantum group parameter $q=e^{i\pi g/(1-g)}$.[34] By analogy, we expect the S matrix for the cyclic spin case to be given (up to the overall factor) by the Yang–Baxter solution for a cyclic spin and a spin $1/2$. This R matrix is an object studied long ago.[35] It is conveniently written as a matrix in $\Pi_1 \otimes \Pi_\delta$,

$$\tau=\begin{pmatrix} w_0 S_0+2w_3 S_3 & 2w_1 S_- \\ 2w_1 S_+ & w_0 S_0-2w_3 S_3 \end{pmatrix} \tag{A.1}$$

where

$$a=\sin(\gamma+u), \qquad b=\sin u, \qquad c=\sin\gamma \tag{A.2}$$

and

$$w_0=\frac{a+b}{2}, \qquad w_3=\frac{a-b}{2}, \qquad w_1=\frac{c}{2}$$

and in the cyclic representation, $S_\pm$ act as indicated above, while

$$S_0\,|m\rangle=\frac{q^m+q^{-m}}{q^{1/2}+q^{-1/2}}$$

$$S_3\,|m\rangle=\frac{q^m-q^{-m}}{2(q^{1/2}-q^{-1/2})}$$

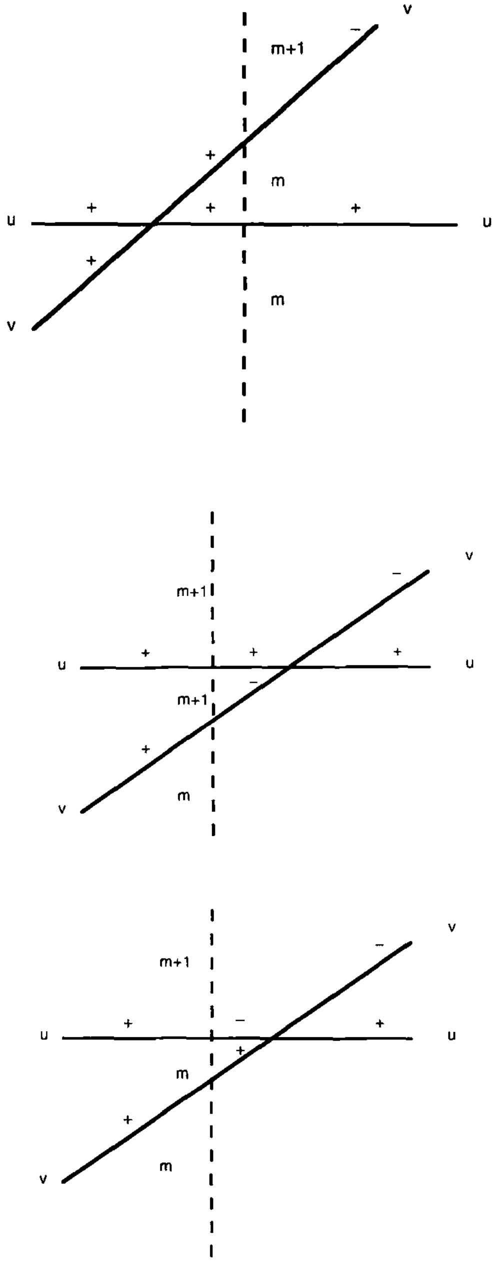


Fig. 4. Configurations involved in checking the YB equation with two spins 1/2 and a cyclic representation.

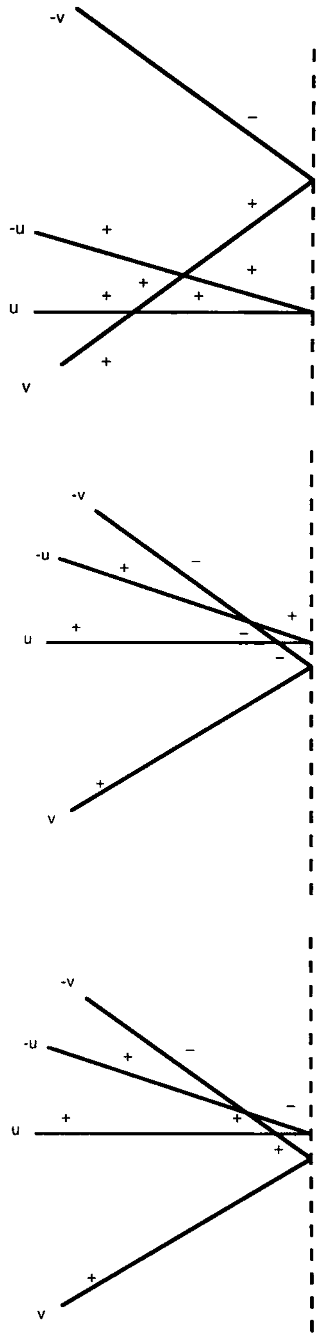


Fig. 5. Configurations involved in checking BYB with no degree of freedom at the boundary.

Setting $u=(\pi/t)(\frac{1}{2}-\delta)+v$ and taking as above the limit $q^\delta=C\gg 1$, one finds the R matrix

$$\begin{aligned} R^{+,m}_{+,m} &= e^{-iv}q^{-m} \\ R^{-,m+1}_{+,m} &= q^{-m} \\ R^{-,m}_{-,m} &= e^{-iv}q^{m} \\ R^{+,m-1}_{-,m} &= q^{m} \end{aligned} \tag{A.3}$$

The result of ref. 21 is that the boundary sine-Gordon S matrix provides a solution of the *boundary* Yang–Baxter equations of the form

$$R=\begin{pmatrix} e^{-iv} & 1 \\ 1 & e^{-iv} \end{pmatrix} \tag{A.4}$$

while (A.3) is a solution of the *ordinary* Yang–Baxter equation. To map the two, it is tempting to simply forget the cyclic degrees of freedom, which appear only as rapidity-independent phases. It is, however, not totally possible, and this has to do with the difference between BYB and YB even for massless particles. Indeed in a massless theory the left–right scattering is rapidity independent, but it might still involve some phases. As such, (A.4) solves BYB, but does *not* solve YB, because of the left–right scattering phases. When considering YB, there are no left–right scattering phases, so the equivalent terms are furnished by the cyclic degrees of freedom in (A.3). This is illustrated in Figs. 4 and 5. The complete translation shows that, if (A.3) satisfies YB, then (A.4) satisfies BYB indeed. In Figs. 4a–4c we consider a particular case of the YB equation involving the scattering of two spins 1/2 and a "cyclic" spin. The weight of the first figure is $W_1=a(u-v)\,e^{-iu}e^{-2im/t}$, that of the second figure $W_2=b(u-v)\,e^{-iu}e^{-i\pi m/t}e^{-i\pi(m+1)/t}$, and that of the third $W_3=ce^{-iv}e^{-2i\pi m/t}$. The fact that YB holds means that $W_1=W_2+W_3$.

In Figs. 5a–5c we consider a particular case of the BYB equation involving a spin 1/2 bouncing off a boundary. The weight of the first figure is $W'1=e^{-iu}a_{LL}(u-v)\,a_{LR}(u-v)$, the weight of the second figure is $W'_2=e^{-iu}b_{LL}(u-v)\,b_{LR}(u-v)$, and the weight of the third $W'_3=e^{-iv}c_{RR}(u-v)\,a_{LR}(u-v)$. The fact that BYB holds means $W'_1=W'_2+W'_3$, which one checks easily since the RR elements are identical to the ones in (A.2) and $a_{LR}=1$, $b_{LR}=e^{-i\gamma}$, $c_{LR}=0$.

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NOTE ADDED IN PROOF

The careful reader might have noticed that the magnetic field in the Kondo problem is coupled to the impurity spin only. This is equivalent to a field coupled to the total spin using results of J. H. Lowenstein, *Phys. Rev. B* **29** (1984), 4120.

REFERENCES

1. N. Andrei, K. Furuya, and J. Lowenstein, *Rev. Mod. Phys.* **55**:331 (1983).
2. A. M. Tsvelick and P. B. Wiegmann, *Adv. Phys.* **32**:453 (1983).
3. C. L. Kane and M. P. A. Fisher, *Phys. Rev. B* **46**:15233 (1992).
4. X. G. Wen, *Phys. Rev. B* **41**:12838 (1990); **43**:11025 (1991).
5. F. P. Milliken, C. P. Umbach, and R. A. Webb, *Solid State Commun* **97**:309 (1996).
6. K. Moon, H. Yi, C. L. Kane, S. M. Girvin, and M. P. A. Fisher, *Phys. Lett.* **71**:4381 (1993) [cond-mat/9304010].
7. P. Fendley, A. W. W. Ludwig, and H. Saleur, *Phys. Rev. Lett.* **74**:3005 (1995) [cond-mat/9408068].
8. A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, *Rev. Mod. Phys.* **59**:1 (1987).
9. S. Chakravarty and J. Rudnick, *Phys. Rev. Lett.* **75**:701 (1995).
10. V. Bazhanov, S. Lukyanov, and A. B. Zamolodchikov, *Comm. Math. Phys.* **177**:381 (1996) [hep-th/9412229].
11. P. Fendley, F. Lesage, and H. Saleur, *J. Stat. Phys.* **79**:799 (1995) [hep-th/9409176].
12. P. Fendley, A. Ludwig, and H. Saleur, *Phys. Rev. B* **52**:8934 (1995) [cond-mat/9503172].
13. K.-D. Schotte, *Z. Phys.* **230**:99 (1970).
14. P. W. Anderson and G. Yuval, *Phys. Rev. Lett.* **23**:89 (1969); G. Yuval and P. W. Anderson, *Phys. Rev. B* **1**:1522 (1970).
15. P. Fendley and H. Saleur, *Phys. Rev. Lett.* **75**:4492 (1995) [cond-mat/9506104].
16. V. Bazhanov, S. Lukyanov, and A. B. Zamolodchikov, [hep-th/9604044] to appear.
17. P. Fendley, H. Saleur, and N. P. Warner, *Nucl. Phys. B* **430**:577 (1994) [hep-th/9406125].
18. P. Fendley, A. Ludwig, and H. Saleur, *Phys. Rev. Lett.* **75**:2196 (1995) [cond-mat/9505031].
19. C. Itzykson, H. Saleur and J. B. Zuber, *Conformal Invariance and Appplications to Statistical Mechanics*, (World Scientific, Singapore, 1988.
20. E. K. Sklyanin, *Funct. Anal. Appl.* **16**:263; **17**:273 (1983).
21. S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys. A9*:3841 (1994) [hep-th/9306002].
22. A. N. Kirillov and N. Yu. Reshetikhin, *J. Phys. A* **20**:1565, 1587 (1987).
23. E. Date, M. Jimbo, K. Miki, and T. Miwa, *Commun. Math. Phys.* **137**:133 (1991).

24. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
25. Al. B. Zamolodchikov, *Phys. Lett. B* **253**:391 (1991); F. Gliozzi and R. Tateo, Thermodynamic Bethe ansatz and threefold triangulations, [hep-th/9505152]; E. Frenkel and A. Szenes, Thermodynamic Bethe ansatz and dilogarithm identities I [hep-th/9506215].
26. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1989).
27. G. E. Andrews, R. J. Baxter and P. J. Forrester, *J. Stat. Phys.* **35**193 (1984).
28. A. M. Tsvelik, *J. Phys. A* **28**:L625 (1995) [cond-mat/9502022].
29. M. Fabrizio and A. Gogolin, *Phys. Rev. B* **50**:17732 (1994).
30. A. M. Tsvelik, *Phys. Rev. B* **52**:4366 (1995).
31. C. Tracy and H. Widom, Proofs of two conjectures related to the thermodynamic Bethe ansatz [solv-int/9509003].
32. K. Leung, R. Egger, and C. Mak, *Phys. Rev. Lett.* **75**:3344 (1995) [cond-mat/9509078].
33. M. P. A. Fisher and W. Zwerger, *Phys. Rev. B* **32**:6190 (1985).
34. P. Fendley, *Phys. Rev. Lett.* **71**: 2485 (1993) [cond-mat/9304031].
35. E. K. Sklyanin, *J. Phys. A* **21**:2375 (1988).